$$\left[\left(-\frac{d}{dt}\frac{\partial E}{\partial q^{\cdot}}+\frac{\partial E}{\partial q}+Q\right)\frac{\partial q}{\partial r}\right]+O\left(\frac{1}{N}\right)$$

$$R_\infty=\left[\left(\frac{d}{dt}\frac{\partial E}{\partial q^{\cdot}}-\frac{\partial E}{\partial q}-Q\right)\frac{\partial q}{\partial r}\right]_\infty+\left[\left(\frac{d}{dt}\frac{\partial E}{\partial x^{\cdot}}-\frac{\partial E}{\partial x}-Z\right)\frac{\partial x}{\partial r}\right]_\infty$$

The subscript ∞ means that the corresponding quantity is calculated along the motion of the system with constraints. The last term in the expression for R_∞ is identically equal to zero, since the condition that the expression within the parenthesis in this term vanishes when $q = q^{\cdot} = q^{\cdot\cdot} = 0$ is Lagrange's equation for motion with constraints. Using (2.1) again we now obtain

$$N\frac{\partial W}{\partial r}+R_\infty=-\left(\frac{d}{dt}\frac{\partial E}{\partial q^{\cdot}}-\frac{\partial E}{\partial q}-Q\right)\frac{\partial q}{\partial r}+\left[\left(\frac{d}{dt}\frac{\partial E}{\partial q^{\cdot}}-\frac{\partial E}{\partial q}-Q\right)\frac{\partial q}{\partial r}\right]_\infty+$$
$$O\left(\frac{1}{N}\right)=A\left(x_\infty\right)q^{\cdot\cdot}\cdot\left(\frac{\partial q}{\partial r}\right)_\infty+\left(N^{-1/2}\right)$$

Integrating from the left and right with respect to t from t_1 to t_2, using integration by parts from the right, and taking into account the fact that $q^{\cdot} = O(N^{-1/2})$, $r_\infty^{\cdot} = O(1)$, we obtain the estimate (2.2) of the theorem.

REFERENCES

1. PAINLEVE P., Lectures on Friction. Gostekhizdat, Moscow, 1954.
2. RUBIN H. and UNGAR P., Motion under a strong constraining force. Communs. Pure and Appl. Math. 10, 1, 1957.
3. TAKENS F., Motion under influence of a strong constraining force. Global theory Dynamic Systems. B.: Springer-Verlag. 1980.
4. VAN KAMPEN N.G., Elimination of fast variables. Phys. Repts. 124, 2, 1985.
5. BENETTIN G., GALGANI L. and GIORGILLI A., Realization of holonomic constraints and freezing of high frequency degrees of freedom in the light of classical perturbation theory. Communs. Math. Phys., 113, 1, 1987.

Translated by L.K.

PMM U.S.S.R.,Vol.54,No.5,pp.708-711,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

THE ASYMPTOTIC STABILIZABILITY OF POSITIONS OF RELATIVE EQUILIBRIUM OF A SATELLITE - GYROSTAT*

V.A. ATANASOV

A theorem proved in /1/ is used to study the possibility of asymptotic stabilization of the equilibrium orientations of a satellite-gyrostat using control moments applied to the rotors.

The asymptotic stabilizability of the stationary motions of mechanical systems with cyclic coordinates was also discussed in /2/, where the sufficient condition of stability was formulated. This, as well as the analogous condition of /1/, follows from the classical theory on the sufficient conditions of stabilization /3/. However, in the theorem in /1/ the condition in question leads, by virtue of taking into account the specific features of the systems with cyclic coordinates, to the study of the rank of a matrix of lower dimensions. From this point of view the theorem in /1/ is more suitable for use when studying the stabilizability of the stationary motions of specific mechanical systems.

**Prikl.Matem.Mekhan.*,54,5,861-863,1990

We shall assume that the centre of mass of the satellite describes a circular orbit in a Newtonian field of force, and consider a restricted formulation of the problem.

Following /4/ we select an inertial system of coordinates $O_1\xi_i$, whose origin coincides with the centre of attraction. We also introduce the following systems: Ox_i, whose axes coincide with the central principal axes of inertia with the origin at the centre of mass, and the orbital coordinate system OX_i, whose X_1 axis is directed along the tangent to the orbit towards the motion, the X_3 axis along the radius vector O_1O, and the X_2 axis supplementing X_1 and X_3 to the right trihedron. We shall define the orientation of the satellite in the orbital system using the aeroplane angles α, β, γ;

$$\alpha_i = \cos(X_1, x_i),\ \beta_i = \cos(X_2, x_i),\ \gamma_i = \cos(X_3, x_i)$$

i.e. the cosines of the angles between the OX_i and Ox_i axes.

We shall assume that the gyrostat has three rotors whose axes are directed along the principal axes of inertia, and φ_i is the angle of rotation of the i-th rotor. We have

$$\omega = \Omega q^{\cdot} + \Omega\omega_0^*$$

$$\omega = \begin{Vmatrix}\omega_1\\ \omega_2\\ \omega_3\end{Vmatrix},\quad q = \begin{Vmatrix}\beta\\ \gamma\\ \alpha\end{Vmatrix},\quad \Omega = \begin{Vmatrix}0 & 1 & \beta_1\\ \sin\gamma & 0 & \beta_2\\ \cos\gamma & 0 & \beta_3\end{Vmatrix},\quad \omega_0^* = \begin{Vmatrix}0\\ 0\\ \omega_0\end{Vmatrix}$$

Here ω is the absolute angular velocity in the system Ox_i, q is the column of generalized coordinates and ω_0 is the orbital angular velocity.

We shall write the kinetic energy of the system in the form

$$T = T_2 + T_1 + T_0,\ T_2 = {}^1/_2 q^{\cdot T}\Omega^T I\Omega q^{\cdot} + \varphi^{\cdot} J\Omega q^{\cdot} + {}^1/_2\varphi^{\cdot} J\varphi^{\cdot}$$
$$T_1 = \omega_0^{*T}\Omega^T I\Omega q^{\cdot} + \omega_0^{*T}\Omega^T I\varphi^{\cdot},\ T_0 = {}^1/_2\omega_0^{*T}\Omega^T I\Omega\omega_0^*$$
$$I = \operatorname{diag}(A_1, A_2, A_3),\ J = \operatorname{diag}(J_1, J_2, J_3),\ \varphi^{\cdot} = [\varphi_1^{\cdot}, \varphi_2^{\cdot}, \varphi_3^{\cdot}]^T$$

where A_i is the i-th principal central moment of inertia of the gyrostat and J_i is the axial moment of inertia of the i-th rotor.

The potential energy is given by the expression

$$\pi = \frac{3}{2}\omega_0^2\sum_{i=1}^{3} A_i\gamma_i^2 + \text{const}$$

We see that the coordinates φ_i are cyclic. Changing to Routh variables, we obtain the following expressions for the components $R = R_2 + R_1 - W$ of the Routh function:

$$R_2 = {}^1/_2 q^{\cdot T}\Omega^T B\Omega q^{\cdot},\ B = \operatorname{diag}(B_1, B_2, B_3),\ B_i = A_i - J_i$$
$$R_1 = g^T q^{\cdot},\ g = [g_1, g_2, g_3]^T = \Omega^T p + \Omega^T B\Omega\omega_0^*$$
$$W = \pi - p^T\Omega\omega_0^* - {}^1/_2\omega_0^*\Omega^T B\Omega\omega_0^* + {}^1/_2 p^T \operatorname{diag}(J_i^{-1})\, p$$

Here $p = [p_1, p_2, p_3]^T$, p_j $(j = 1, 2, 3)$ are the cyclic moments.

The positions of relative equilibrium of the gyrostat $q^0 = [\beta_0, \gamma_0, \alpha_0]^T$, $p = p^0$, $p_j = \text{const}$ $(j = 1, 2, 3)$ are the stationary motions of the system, and are given by the equations

$$\left.\frac{\partial W}{\partial\beta}\right|_0 = 3\omega_0^2\sum_{i=1}^{3} A_i\gamma_i\frac{\partial\gamma_i}{\partial\beta} - \omega_0^2\sum_{i=1}^{3} B_i\beta_i\frac{\partial\beta_i}{\partial\beta} - \omega_0\sum_{i=1}^{3} p_i\frac{\partial\beta_i}{\partial\beta} = 0$$
$$\left.\frac{\partial W}{\partial\gamma}\right|_0 = 3\omega_0^2(A_2 - A_3)\gamma_2\gamma_3 - \omega_0^2(B_2 - B_3)\beta_2\beta_3 - \omega_0(p_2\beta_3 - p_3\beta_2) = 0$$
$$\left.\frac{\partial W}{\partial\alpha}\right|_0 = -3\omega_0^2\sum_{i=1}^{3} A_i\alpha_i\gamma_i = 0$$

The last equation determines the manifold of positions of relative equilibrium of the gyrostat, and the first two equations give the values of the cyclic moments for which the given equilibrium orientation is realized.

We shall consider the feasibility of asymptotic stabilization of the equilibrium orientations belonging to the following classes of positions of relative equilibrium of the gyrostat.

Class A. One of the principal axes of inertia is directed along the tangent to the orbit, and the remaining two make a constant angle γ_0 with the axes X_2 and X_3 respectively. With this orientation $q^0 = [0, \gamma_0, 0]^T$, $\gamma_0 \in [0, 2\pi)$, and the values of the cyclic moments $p^0 = [p_1, p_2, p_3]^T$, for which they are realized are given by the relations

$$p_1 = 0;\ [3\omega_0 (A_2 - A_3) + \omega_0 (B_2 - B_3) \sin \gamma_0 \cos \gamma_0 + p_2 \sin \gamma_0 + p_3 \cos \gamma_0 = 0$$

Class B. One of the principal axes of inertia is directed along the radius vector X_3, and the other two make a constant angle β_0 with X_1 and X_2 respectively. With this orientation $q^0 = [\beta_0, 0, 0]^T$, $\alpha_0 \in (-\pi/2, \pi/2)$, and the values of the cyclic moments $p^0 = [p_1, p_2, p_3]^T$ must satisfy the relations

$$p_3 = 0;\ \omega_0 (B_2 - B_1) \sin \beta_0 \cos \beta_0 = p_1 \cos \beta_0 - p_2 \sin \beta_0$$

In both cases $\xi = [\xi_1, \xi_2, \xi_3]^T$ and $\eta = [\eta_1, \eta_2, \eta_3]^T$ are, respectively, the perturbation in position coordinates and cyclic moments from their stationary values, and we make the following change of variables: $\xi = \Omega_0^{-1}x;\ \Omega_0 = \Omega\,|_{q=q^0}$. To a first approxiation the equations of perturbed motion in the neighbourhood of the stationary motion $q = q^0,\ p = p^0$ are obtained in the form

$$Bx^{\cdot\cdot} = -C^*x + G^*x^{\cdot} - N^*\eta - Eu;\ \eta^{\cdot} = u \tag{1}$$

Here

$$C^* = \Omega_0^{-1T} C\Omega_0^{-1},\ G^* = \Omega_0^{-1T} G\Omega_0^{-1},\ N^* = \Omega_0^{-1T} N$$
$$C = \| c_{ij} \|_{i,\,j=1}^3,\ c_{ij} = \partial^2 W (q^0, p^0) / \partial q_i \partial q_j$$
$$G = \| \gamma_{ij} \|_{i,\,j=1}^3,\ \gamma_{ij} = \partial g_j (q^0, p^0) / \partial q_i - \partial g_i (q^0, p^0) / \partial q_j$$
$$N = \| n_{ij} \|_{i,\,j=1}^3,\ n_{ij} = \partial^2 W (q^0, p^0) / \partial q_i \partial p_j$$

E is the unit (3×3) matrix and $u = [u_1, u_2, u_3]^T$ are the moments applied to the rotors.

Formally, Eqs.(1) do not differ from the linearized equations of perturbed motion in /1/. Therefore the condition of stabilizability formulated there can be formally transferred to the mechanical system discussed here, irrespective of the fact that the kinetic energy is not a quadratic form of the generalized velocities only. Thus we can assert the following: the equilibrium orientations discussed here will be asymptotically stabilizable relative to position coordinates and all velocities with the help of moments applied to the rotors, provided that the following condition holds:

$$\operatorname{rank} S = 6,\quad S = \| Q, PQ, \ldots, P^5Q \| \tag{2}$$
$$Q = \begin{Vmatrix} E \\ N^* \end{Vmatrix},\quad P = \begin{Vmatrix} G^*B^{-1} & E \\ -C^*B^{-1} & 0 \end{Vmatrix}$$

(in this notation there was no previous passage to normal coordinates).

Carrying out elementary algebra and direct computations we obtain, for the equilibrium orientations of class A, the following two sixth-order minors M_1 and M_2 of the matrix S:

$$M_1 = D\,\frac{A_1 - A_3}{B_2} \cos \gamma_0,\quad M_2 = D\,\frac{A_1 - A_2}{B_3} \sin \gamma_0 \tag{3}$$
$$D = 27\omega_0^7 \left(\frac{A_2 - A_3}{B_1}\right)^2 \cos^2 2\gamma_0$$

They cannot be simultaneously equal to zero, except in the cases when $\gamma_0 = \pi/4,\ 3\pi/4$. Therefore, all orientations belonging to class A, except the ones for which $\gamma_0 = \pi/4,\ 3\pi/4$, can be made asymptotically stable relative to the position coordinates and all moments, using moments applied to the rotors.

In the case of equilibrium orientations for which $\gamma_0 = \pi/4,\ 3\pi/4$, we can show that the rank $s < 6$, and therefore /5/ the possibility of stabilization is determined, in this case, by terms of higher order of smallness.

In the case of equilibrium orientations of class B, we can find a sixth-order minor different from M_1 in (3) by replacing $\cos^2 2\gamma_0 \cos \gamma_0$ by $\cos \beta_0$. The minor is always non-zero since $\beta_0 = \pm\pi/2$ is the critical value of the aeroplane angles. Therefore, all orientations of class B are asymptotically stabilizable relative to position coordinates and all moments, using moments applied to the rotors.

The possibility of asymptotic stabilization of the equilibrium orientations discussed here was proved in /5/ only in relation to positional coordinates and position velocities.

This work was sponsored by the Science Committee of NRB.

REFERENCES

1. ATANASOV V.A. and LILOV L.K., On the stabilizability of steady motions of systems with pseudocyclic coordinates. PMM, 52, 5, 1988.
2. KRASINSKII A.YA. and RONZHIN V.V., On stabilization of steady motions of mechanical systems with cyclic coordinates. PMM, 52, 4, 1988.

3. KRASOVSKII N.N., Problems of stabilization of controlled motions. in: Melkin I.G., Theory of Stability of Motion. Nauka, Moscow, 1966.
4. ANCHEV A., Equilibrium Orientations of a Satellite-Gyrostat. Izd. BAN, Sofiya, 1982.
5. LILOV L.K., Stabilization of the steady-state motions of mechanical systems with respect to some of the variables. PMM, 36, 6, 1972.

Translated by L.K.

PMM U.S.S.R.,Vol.54,No.5,pp.711-713,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

THE MINIMUM DIMENSIONS OF THE CONTROL VECTOR IN THE LINEAR DYNAMIC PROBLEM OF STABILIZATION*

V.N. SOKOLOV

A mathematical formalization is proposed of the problem of estimating the number of engines necessary to stabilize a mechanical elastic system, functioning in conditions of zero gravity, in a specified position. Conditions are given which allow the class of control matrices imparting the property of full controllability to dynamic systems to be described /1/. The analysis of conditions of full controllability for mechanical systems in the neighbourhood of the position of equilibrium was given in /2/.

We consider the following dynamic selfsimilar system:

$$x^{\cdot} = Fx + Gu, \quad x \in R^n, \quad u \in R^m \tag{1}$$

where F and G are constant matrices, x is the state vector and u the control vector. We know /1, 2/ that if the condition of full controllability

$$\operatorname{rank} \| G, \quad FG, \quad F^2G, \ldots, F^{n-1}G \| = n \tag{2}$$

holds, then a control $u(t)$ exists which takes the system (1) from any initial position x_0 to the origin of coordinates. If condition (2) does not hold, then such a control does not, in general, exist. Our aim is to determine the minimum number of scalar control functions u_i, i.e. the minimum dimensions of the control vector for which the condition of full controllability can be attained by a suitable choice of the control matrix G. The answer to this problem is given by the following theorem.

Theorem. Let k_i be the number of linearly independent eigenvectors corresponding to the i-th eigenvalue of the matrix F, and $k = \max_i k_i$. Then k will be the minimum dimension of the control vector $u(t)$ for which the choice of the matrix G can still result in satisfying the condition of complete controllability (2).

Following /3/, we shall introduce a number of concepts and assertions. We shall call the vector g the root vector corresponding to the eigenvalue λ_i, provided that

$$(F - \lambda_i E)^h g = 0 \tag{3}$$

for some integral value of $h > 0$. We shall call the heigt j of the vector g the smallest value of h for which condition (3) holds, i.e. $(F - \lambda_i E)^{j-1} g \neq 0$ and $(F - \lambda_i E)^j g = 0$. The zero vector has zero height by definition. The set of root vectors corresponding to some eigenvalue λ_i, forms a root subspace P_i, invariant under the transformation $F - \lambda_i E$, and consequently also invariant under the operator F. The root subspace P_i in turn decomposes into k_i cyclic subspaces (k_i is the number of linearly independent eigenvectors corresponding to the i-th eigenvalue), invariant under the operator F. These subspaces Π_μ^i, $\mu = 1, 2, \ldots k_i$, are stretched over the vectors $g_{\mu\nu}^i$, which satisfy the condition

Prikl.Matem.Mekhan.,54,5,864-866,1990